

Young's lattice and dihedral symmetries revisited: Möbius strips & metric geometry

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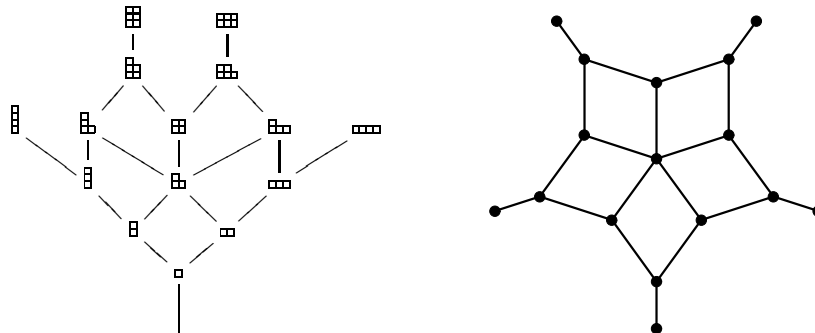
Abstract

A cascade of dihedral symmetries is hidden in Young's lattice of integer partitions. In fact, for each $N \in \mathbb{Z}_{\geq 3}$ the Hasse graph of the subposet \mathbb{Y}_N consisting of the partitions with maximal hook length strictly less than N has the dihedral group of order $2N$ as its symmetry group. Here a new interpretation of those Hasse graphs is presented, namely as the 1-skeleta of the injective hulls of certain finite metric spaces.

1 Introduction and some history

For each positive integer N let \mathbb{Y}_N denote the set of those integer partitions whose maximal hook lengths are strictly less than N . Recall that a partition λ covers a partition μ in Young's lattice means that the Young diagram of μ is got by removing an inner corner box from the Young diagram of λ . Consider the Hasse diagram of the subposet of Young's lattice on \mathbb{Y}_N . Let us call $\text{Hasse}(\mathbb{Y}_N)$ the underlying undirected abstract graph.

Example (Hasse diagram of \mathbb{Y}_5 and its undirected Hasse graph)



This graph has a 5-fold (cyclic) symmetry, and its full symmetry group is a dihedral group of order 10.

The phenomenon generalizes to all N .

Theorem 1.1 (see [Su1, Theorem 2.1]) *For each positive integer N the cyclic group of order N acts faithfully on $\text{Hasse}(\mathbb{Y}_N)$.*

Remark 1.2 Mapping a partition to its dual (also known as conjugate or transpose) partition restricts to an involution $\mathbb{Y}_N \rightarrow \mathbb{Y}_N$, which is different from the identity if $N \geq 3$. Together with the cyclic symmetry of order N it generates the full symmetry group $\text{Aut}(\text{Hasse}(\mathbb{Y}_N))$, which is a dihedral group of order $2N$ if $N \geq 3$.

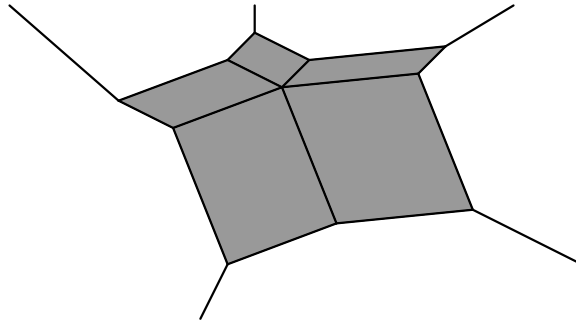
The cyclic symmetries emerged as a byproduct of the work on abelian ideals in a Borel subalgebra \mathfrak{b} of a finite-dimensional complex simple Lie algebra \mathfrak{g} (see [Su2]), namely when \mathfrak{g} has type A_{N-1} .

Here is a quick recapitulation. Let us use the convention that the nilradical of \mathfrak{b} is the sum of the root spaces \mathfrak{g}_φ where φ runs over the *positive* roots. Then each abelian ideal $\mathfrak{a} \trianglelefteq \mathfrak{b}$ is a sum of root spaces $\mathfrak{a} = \bigoplus_{\varphi \in \Psi} \mathfrak{g}_\varphi$ for a certain subset Ψ of the set of positive roots. If ρ denotes half the sum of the positive roots, then $\rho + \sum_{\varphi \in \Psi} \varphi$ is the unique integral weight in the interior of an affine Weyl group translate of the fundamental alcove (for instance $\rho + \theta$, where θ is the highest root, is such an integral weight in the interior of an alcove, which is adjacent to the fundamental alcove). The union of all those alcoves that one gets if \mathfrak{a} runs over all abelian ideals in \mathfrak{b} , is exactly the fundamental alcove dilated by a factor of two.

In the special case of $\mathfrak{sl}_N(\mathbb{C})$ the N -fold cyclic symmetry of the fundamental alcove, which is reflected in the fact that the affine Dynkin diagram is an N -cycle, yields the cyclic symmetry acting on \mathbb{Y}_N after identifying elements of \mathbb{Y}_N in an evident way with abelian ideals and hence with those alcoves that lie in (and tessellate) the fundamental alcove dilated by a factor of two.

Since integer partitions and Young's lattice are so basic mathematical objects, it was natural to look for an independent approach to the cyclic symmetries. This was realized in [Su1], and I reported on it in a talk "A surprising result about Young's lattice" in the ETH Zurich Algebra-Topology Seminar in October 2001. In October 2007, I mentioned the cyclic symmetries parenthetically in a talk "Some insights from cluster categories" at INdAM in Rome at a workshop organized by Paolo Papi and Eric Sommers. For recent developments see [BZ, TW].

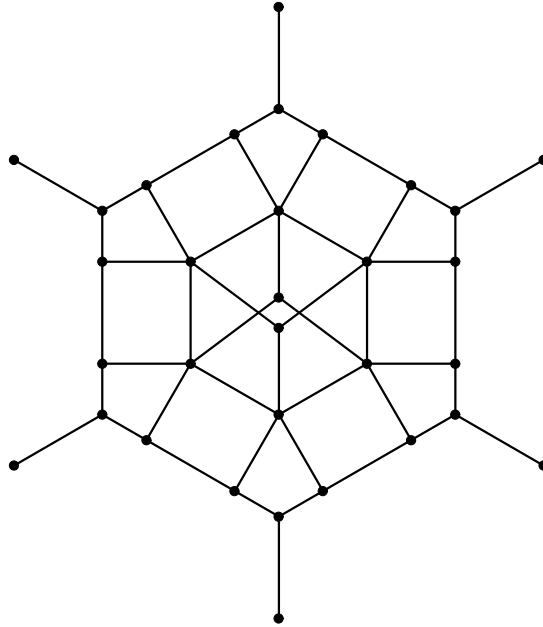
On February 29, 2012, Urs Lang gave a talk "Injective hulls of metric spaces: old and new" in the ETH Zurich Geometry Seminar. In the introduction he showed the picture [Dr1, top of p. 338], which roughly looks as follows



and shows the injective hull (of one of three possible types) of a generic metric space with five points.

A drawing for the injective hull of a generic metric space with four points shows a rectangle with four “antennas” attached, one at each vertex; for three points one gets a Y-shaped tree; for two points the injective hull is a line segment; and the injective hull of a one-point space has one point. Hence for $N \in \{1, 2, 3, 4, 5\}$ we recognize the geometric realizations of the graphs $\text{Hasse}(\mathbb{Y}_N)$ as the 1-skeleta of the injective hulls of certain N -point metric spaces.

What about the $N = 6$ situation? From [Su1] the graph $\text{Hasse}(\mathbb{Y}_6)$ on 32 vertices and with 48 edges looks as in the following picture.



This graph is indeed isomorphic to the 1-skeleton of the injective hull of the six-point metric space visualized in [HKM, Fig. 1 on p. 570] or in [HJ, Fig. 2 on p. 176].

In general, $\text{Hasse}(\mathbb{Y}_N)$ can be geometrically realized as the 1-skeleton of the injective hull of an N -point metric space where the N points form an orbit under an isometric cyclic action.

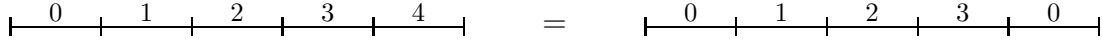
Theorem 1.3 *Let $N \in \mathbb{Z}_{\geq 2}$. Consider a geometric realization of the graph $\text{Hasse}(\mathbb{Y}_N)$ as a metric space in which all edges have length 1 and let X_N be its boundary, that is, X_N is the N -point subspace consisting of the N pending vertices (the tips of the “antennas”) corresponding to the rectangular partitions (j^{N-j}) (for $j = 1, \dots, N-1$) together with the empty partition. Then the 1-skeleton of the injective hull $E(X_N)$ is a geometric realization of the graph $\text{Hasse}(\mathbb{Y}_N)$.*

For a proof see Remark 3.14.

2 Möbius strips and Young diagrams

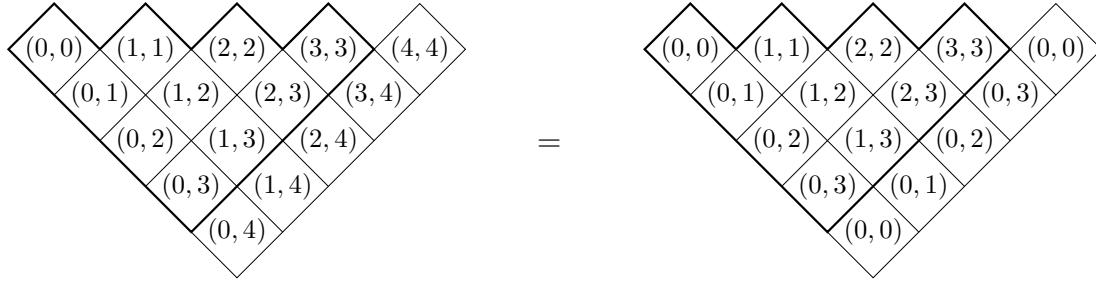
Let $N \in \mathbb{Z}_{\geq 2}$. We consider the discrete circle with N sites $X = X_N = \{0, 1, \dots, N-1\}$. Two sites $j, k \in X$ are neighbouring if $|k - j| = 1$ or $|k - j| = N - 1$. Expressed more uniformly, the two sites j and $j + 1$ are neighbouring for $j = 0, \dots, N - 1$, where N is identified with 0.

Example The circle for $N = 4$ with its 4 sites.



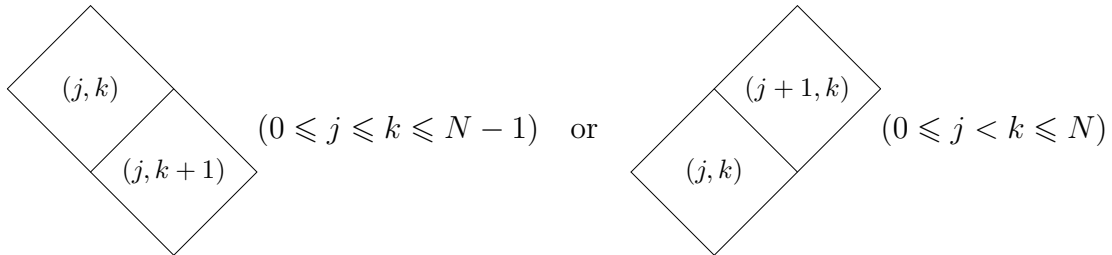
Next we consider a discrete Möbius strip $\mathfrak{X} = \mathfrak{X}_N$ with X embedded as its boundary. Namely, $\mathfrak{X} = X \times X / \{(j, k) \sim (k, j) \mid j, k \in X\}$. The Möbius strip \mathfrak{X} has $\frac{1}{2}N(N+1)$ sites represented by (j, k) ($0 \leq j \leq k \leq N-1$), and we may extend this to $0 \leq j \leq k \leq N$ by identifying N with 0. The embedding $X \hookrightarrow \mathfrak{X}$ is $j \mapsto (j, j)$. By convention and slightly abusing the notation, we write $(j, k) \in \mathfrak{X}$ for the *class* of the pair (j, k) (i.e., its site in \mathfrak{X}) and hence $(j, k) = (k, j) \in \mathfrak{X}$. Some definitions below (for instance, in Lemma 2.4) use the independence under the exchange $j \leftrightarrow k$ for their well-definedness, something that we keep in mind with the tacit understanding.

Example The Möbius strip for $N = 4$ with its 10 sites.



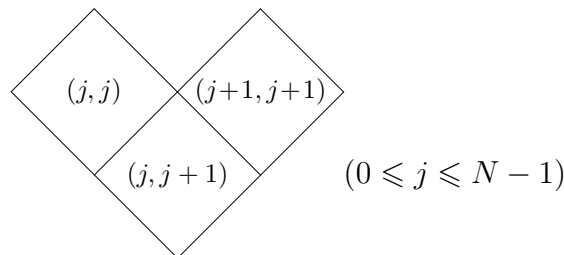
We need to familiarize ourselves with some notations for the geography of the Möbius strip \mathfrak{X} . Let us first have a local inspection.

- 1) Each pair of adjacent sites in the Möbius strip is represented (in at least one way) as

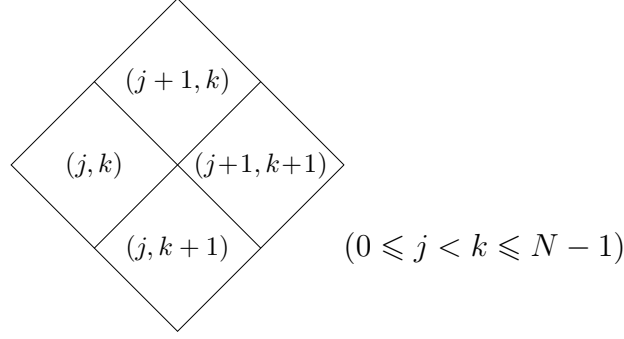


The pairs on the right for $k = N$ are already represented by the pairs of sites $(0, j)$ and $(0, j+1)$ depicted on the left.

- 2) Each triple of sites consisting of a site adjacent to two sites at the boundary of the Möbius strip can be represented as



3) Each 2×2 square of sites in the Möbius strip can be represented as



(For $N = 2$ this square has only three different sites.)

As regards global objects in the geography of \mathfrak{X} , it will soon become evident that homotopically nontrivial loops are important to consider. Let (j_0, k_0) adjacent to (j_1, k_1) adjacent to (j_2, k_2) adjacent to \dots adjacent to (j_{N-1}, k_{N-1}) adjacent to $(j_N, k_N) = (j_0, k_0)$ be such a loop consisting of N sites (see (2.1) for an example).

Definition 2.1 Let

$$\Omega = \Omega_N = \left\{ \mathcal{L} \subseteq \mathfrak{X} \left| \begin{array}{l} |\mathcal{L}| = N \text{ and the sites in } \mathcal{L} \text{ realize a homo-} \\ \text{topically nontrivial loop in the Möbius strip} \end{array} \right. \right\}.$$

Note that for $\mathcal{L} \subseteq \mathfrak{X}$ with $|\mathcal{L}| < N$ the sites in \mathcal{L} cannot realize a homotopically nontrivial loop in the Möbius strip.

Remark 2.2 Since for $\mathcal{L} \in \Omega$ the sites in \mathcal{L} realize a homotopically nontrivial loop, the following property holds: for each $\mathcal{L} \in \Omega$

$$X = \bigcup_{(j,k) \in \mathcal{L}} \{j, k\}.$$

Lemma 2.4 below gives more precise information.

Proposition 2.3 *There is a bijection*

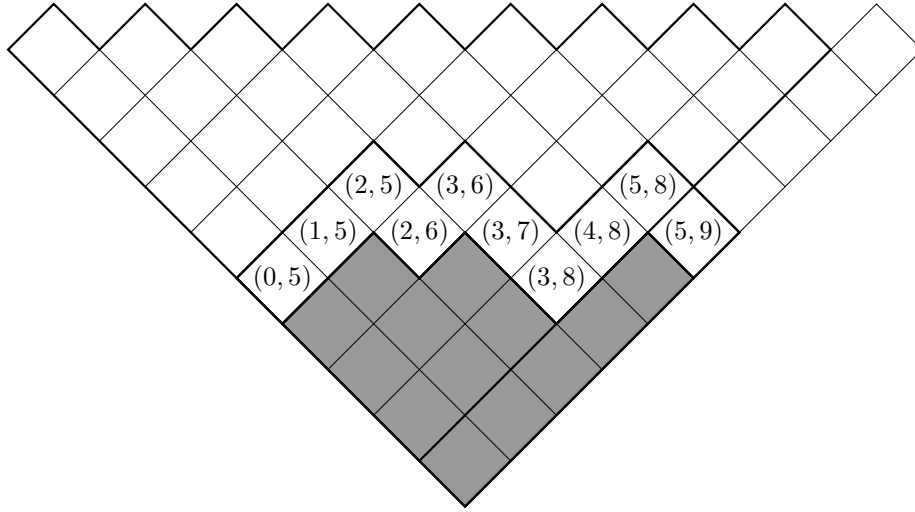
$$\begin{array}{l} \Omega \longrightarrow \mathbb{Y}_N = \{\lambda \mid \lambda \text{ is a partition with maximal hook length} < N\}. \\ \mathcal{L}_\lambda \longleftarrow \lambda \end{array}$$

In particular, $|\Omega| = 2^{N-1}$. The loop \mathcal{L}_λ will be referred to as the (loop associated with the) outer rim of λ .

Proof. Represent the sites of \mathfrak{X} in a triangular shape with sites (j, k) ($0 \leq j \leq k \leq N$). The three corners $(0, 0)$ “upper left”, $(0, N)$ “bottom”, and (N, N) “upper right” of this triangular shape all represent the same site $(0, 0) \in \mathfrak{X}$, and $(0, k) = (k, N) \in \mathfrak{X}$.

For $\mathcal{L} \in \Omega$ we obtain the corresponding partition λ such that the sites below the sites contained in \mathcal{L} (in the triangular shape) are just the boxes of the Young diagram (drawn in Russian convention) of λ . So \mathcal{L} can be considered as the outer rim of λ . (If λ has maximal hook length $h < N - 1$, we could consider λ as having its nonzero parts and in addition $N - 1 - h$ zero parts.)

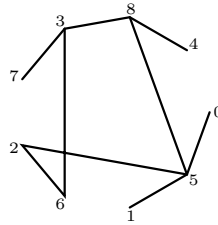
Example Here is an example with $N = 9$ for the partition $\lambda = (5, 3, 3, 2)$.



The outer rim of λ is

$$\mathcal{L}_\lambda = \{(0, 5), (1, 5), (2, 5), (2, 6), (3, 6), (3, 7), (3, 8), (4, 8), (5, 8)\} \subseteq \mathfrak{X}. \quad (2.1)$$

Here is an encoding of \mathcal{L}_λ in terms of a graph with vertex set $X = \{0, \dots, N-1\}$: there is an edge between j and k if and only if $(j, k) \in \mathcal{L}_\lambda$.



Note that in this way we get a connected graph with N vertices and N edges, hence with exactly one cycle (which can be a graph-theoretic loop). From this graph we can easily read off a bijection $\rho : X \rightarrow \mathcal{L}_\lambda$ such that if $\rho(l) = (j, k)$, then $l \in \{j, k\}$, so that Remark 2.2 follows as a corollary. In the example above there is no choice for $\rho(0) = (0, 5)$, $\rho(1) = (1, 5)$, $\rho(7) = (3, 7)$, $\rho(4) = (4, 8)$; for the vertices that belong to the cycle $(5, 2, 6, 3, 8)$, there are exactly two choices:

l	5	2	6	3	8
$\rho(l)$	(2, 5)	(2, 6)	(3, 6)	(3, 8)	(5, 8)
or					
$\rho(l)$	(5, 8)	(2, 5)	(2, 6)	(3, 6)	(3, 8)

For general \mathcal{L}_λ there are always two choices for the bijection ρ except if the associated graph has a graph-theoretic loop, in which case there is a unique such bijection ρ . Note also that the cycle always has odd length. In fact, the edges of the cycle correspond to the sites where the loop \mathcal{L}_λ turns, and the number of those sites is the number of outer corners (if the maximal hook length of λ is strictly less than $N-1$) or the number of outer corners minus 2 (if the maximal hook length of λ is $N-1$) plus the number of inner corners of λ . The result then follows from

$$\#(\text{outer corners of } \lambda) - \#(\text{inner corners of } \lambda) = 1.$$

In the example the outer corners of λ are at $(2, 6)$ and $(3, 8)$ as well as at $(0, 5)$ and $(5, 9)$ (as a site in \mathfrak{X} we have $(0, 5) = (5, 9)$, and \mathcal{L}_λ does not turn there); and the inner corners of λ are at $(1, 6) [\leftrightarrow (2, 5)]$, $(2, 7) [\leftrightarrow (3, 6)]$, and $(4, 9) [\leftrightarrow (5, 8)]$, where $[\leftrightarrow (j+1, k)]$ marks the corresponding site where \mathcal{L}_λ turns around the inner corner $(j, k+1)$ of λ .

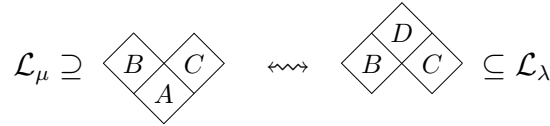
Let us complete the proof of Proposition 2.3. In general, to enumerate all elements \mathcal{L} of Ω , we count separately for $1 \leq k \leq N$ those \mathcal{L} that contain both $(0, k)$ and $(1, k)$. There are $\binom{N-1}{k-1}$ chains of length N consisting of successively adjacent sites starting at $(1, k)$ and ending at (k, N) (all inside the triangular shape). A summation over k gives a total of 2^{N-1} possibilities. \square

Lemma 2.4 *For each site $L = (j, k) \in \mathfrak{X}$ let $e_L := e_j + e_k \in \mathbb{R}^X$ with $e_i(l) := \delta_{i,l}$. Or in other words, the vector $(e_L(l))_{l \in X}$ has entries 1 at positions j and k if $j \neq k$ or has an entry 2 at position j if $j = k \in X$, and all other entries 0. For each $\mathcal{L}_\lambda \in \Omega$ consider the $N \times N$ matrix*

$$T_\lambda := (e_L(l))_{L \in \mathcal{L}_\lambda, l \in X}.$$

Then T_λ is regular, in fact, its determinant is ± 2 .

Proof. We start with $\mathcal{L}_0 = \{(0, k) \mid 0 \leq k < N\}$. The matrix $T_0 = (e_{(0,k)}(l))_{k, l \in X}$ is triangular with its $(0, 0)$ diagonal entry 2 and the other diagonal entries 1, hence with determinant 2. We proceed by induction. If $\lambda \neq ()$, then λ covers a partition μ with outer rim \mathcal{L}_μ . The situation can be depicted as follows:



where $A = (j, k+1)$, $B = (j, k)$, $C = (j+1, k+1)$, and $D = (j+1, k)$. The Young diagram of λ is got by adding the box at A to the Young diagram of μ . We can write $\mathcal{L}_\lambda = (\mathcal{L}_\mu - \{A\}) \cup \{D\}$. Since $e_D = -e_A + e_B + e_C$, the determinants of the matrices T_λ and T_μ are equal up to a sign. (Using the fact that there are exactly one or two distinguished bijections $\rho : X \rightarrow \mathcal{L}_\lambda$ (with $l \in \{j, k\}$ if $\rho(l) = (j, k)$) and that if there are two such bijections, then they differ only by an odd-length cyclic (hence even) permutation, we could remove the ambiguity in the sign of the determinant.) \square

Remark 2.5 The cyclic action that is given by translation in the Möbius strip $\mathfrak{X} \rightarrow \mathfrak{X}$, $(j, k) \mapsto (j+1, k+1)$ induces an action on Ω and hence on the set of partitions with maximal hook length $< N$. This corresponds to the diagonal sliding operation described in [Su1] and will be recalled later (see Theorem 3.6).

3 Injective hulls of finite metric spaces

A metric space Z is called *injective* if every 1-Lipschitz (i. e., distance non-increasing) map $f : A \rightarrow Z$ from a subspace A of any metric space X can be extended to a 1-Lipschitz map $\bar{f} : X \rightarrow Z$.

It was first proved by Isbell [Is] that for every metric space X there is an injective metric space E such that X embeds isometrically into E ; and given such an embedding $e : X \rightarrow E$, there is a unique smallest injective subspace $E(X) \subseteq E$ containing $e(X)$. Let

$e = \overline{\text{id}_{E(X)}} \circ e$ where $\overline{\text{id}_{E(X)}} : E \rightarrow E(X)$ extends $\text{id}_{E(X)}$ as a 1-Lipschitz map. This space $E(X)$ (or more properly the isometric embedding $e : X \rightarrow E(X)$) is called an *injective hull* of X . If $e' : X \rightarrow E'(X)$ is another injective hull of X , then there is an isometry $\iota : E(X) \rightarrow E'(X)$ with $e' = \iota \circ e$.

The injective hull $E(X)$ of a finite metric space X with metric d can be realized as the polyhedral complex that consists of the bounded faces of the polyhedron

$$\Delta(X) := \{f \in \mathbb{R}^X \mid \forall x, y \in X : f(x) + f(y) \geq d(x, y)\} \quad (3.1)$$

and can be shown to be (see [Dr2, Lemma 1])

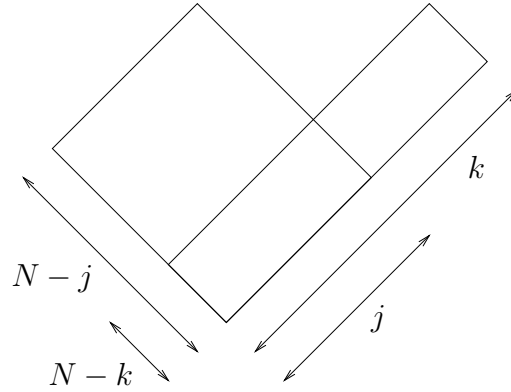
$$E(X) := \{f \in \Delta(X) \mid \forall x \in X \exists y \in X : f(x) + f(y) = d(x, y)\}. \quad (3.2)$$

The distance between two functions $f, g \in E(X) \subseteq \mathbb{R}^X$ is $\|f - g\|_\infty = \max_{x \in X} |f(x) - g(x)|$, and X embeds into $E(X)$ by $x \mapsto e(x) = d(x, \cdot)$.

$$X = X_N = \{0, 1, \dots, N-1\}$$

We consider the metric space $X = X_N = \{0, 1, \dots, N-1\}$ with metric $d(0, j) = j(N-j)$ (for $j = 0, \dots, N-1$; the formula holds also if we insert $j = N$ and identify N with 0) and extended cyclically, that is, $d(j, k) = |k - j|(N - |k - j|)$.

Note that if we let 0 correspond to the empty partition and $j \in X - \{0\}$ to the rectangular partition (j^{N-j}) , then this is the metric induced from the Hasse graph (all edges of length 1) of Young's lattice. In fact, starting with the Young diagram of the partition (j^{N-j}) , we successively remove boxes till we get (j^{N-k}) (for $k > j$) and then successively add boxes till we obtain (k^{N-k}) .



The total number of required moves is

$$\underbrace{j((N-j) - (N-k))}_{\text{remove boxes}} + \underbrace{(k-j)(N-k)}_{\text{add boxes}} = (k-j)(N - (k-j)) = d(j, k).$$

Definition 3.1 (Extension from X to \mathfrak{X} : $f \mapsto \tilde{f}$) We embed $\mathbb{R}^X \hookrightarrow \mathbb{R}^{\mathfrak{X}}$ by

$$f \mapsto \tilde{f} : (j, k) \mapsto \frac{1}{2}(f(j) + f(k) - d(j, k)).$$

This is of course well-defined and $f(j) = \tilde{f}(j, j)$. Note also that the extension commutes with affine combinations, that is, for $(f_i)_{i \in I} \subseteq \mathbb{R}^X$ and $(a_i)_{i \in I} \subseteq \mathbb{R}$ with $\sum_{i \in I} a_i = 1$ we have

$$\left(\sum_{i \in I} a_i f_i \right)^\sim = \sum_{i \in I} a_i \tilde{f}_i.$$

This applies in particular to convex combinations, where all the coefficients a_i are non-negative and sum up to 1.

Let us rewrite the definitions (3.1) and (3.2) as in the following definition.

Definition 3.2 (Injective hull of X)

$$\begin{aligned}\Delta(X) &:= \{f \in \mathbb{R}^X \mid \forall L \in \mathfrak{X} : \tilde{f}(L) \geq 0\}, \\ E(X) &:= \{f \in \Delta(X) \mid \forall j \in X \exists L = (j, k) = (k, j) \in \mathfrak{X} : \tilde{f}(L) = 0\}.\end{aligned}$$

Lemma 3.3 *Let $f \in \mathbb{R}^X$ and let $\tilde{f} \in \mathbb{R}^{\mathfrak{X}}$ be its extension to \mathfrak{X} as in Definition 3.1.*

For any 2×2 square of sites $\begin{array}{c} \diagup \quad \diagdown \\ B \quad C \\ \diagdown \quad \diagup \\ A \end{array} \subseteq \mathfrak{X}$ we have

$$\tilde{f}(A) + \tilde{f}(D) - \tilde{f}(B) - \tilde{f}(C) = 1.$$

For any triple of sites $\begin{array}{c} \diagup \quad \diagdown \\ B \quad C \\ \diagdown \quad \diagup \\ A \end{array} \subseteq \mathfrak{X}$ with B and C at the boundary we have

$$2\tilde{f}(A) + N - \tilde{f}(B) - \tilde{f}(C) = 1.$$

Proof. Recall from Section 2 how to represent the sites in such triples and quadruples. For the triples let $A = (j, j+1)$, $B = (j, j)$, $C = (j+1, j+1)$ and compute

$$\begin{aligned}2\tilde{f}(A) + N - \tilde{f}(B) - \tilde{f}(C) \\ = f(j) + f(j+1) - (N-1) + N - f(j) - f(j+1) = 1\end{aligned}$$

and for the quadruples let $A = (j, k+1)$, $B = (j, k)$, $C = (j+1, k+1)$, $D = (j+1, k)$ and compute

$$\begin{aligned}\tilde{f}(A) + \tilde{f}(D) - \tilde{f}(B) - \tilde{f}(C) \\ = \frac{1}{2}(f(j) + f(k+1) - (k-j+1)(N-k+j-1)) \\ + \frac{1}{2}(f(j+1) + f(k) - (k-j-1)(N-k+j+1)) \\ - \frac{1}{2}(f(j) + f(k) - (k-j)(N-k+j)) \\ - \frac{1}{2}(f(j+1) + f(k+1) - (k-j)(N-k+j)) = 1.\end{aligned}$$

□

Definition 3.4 For a site $L = (j, k) \in \mathfrak{X}$ consider the affine hyperplane

$$H_L := \{f \in \mathbb{R}^X \mid f(j) + f(k) = d(j, k)\} = \{f \in \mathbb{R}^X \mid \tilde{f}(L) = 0\}$$

with \tilde{f} the extension of f to \mathfrak{X} as in Definition 3.1.

For a set of sites $\mathcal{L} \subseteq \mathfrak{X}$ consider the intersection of the affine hyperplanes

$$H_{\mathcal{L}} := \bigcap_{L \in \mathcal{L}} H_L \subseteq \mathbb{R}^X.$$

The next proposition will be superseded by Theorem 3.6.

Proposition 3.5 (Empty partition) *Let $f_0 := d(0, \cdot) \in \mathbb{R}^X$ and let $\mathcal{L}_0 \in \Omega$ be the outer rim (see Proposition 2.3) of the empty partition $()$. Then*

- $H_{\mathcal{L}_0} = \{f_0\}$
- For $L \in \mathfrak{X}$, say $L = (j, k)$ with $0 \leq j \leq k < N$ (or even $0 \leq j \leq k \leq N$), we have $\tilde{f}_0(L) = \tilde{f}_0(j, k) = j(N - k)$
- $f_0 \in E(X)$

Proof. We have $f_0(l) = l(N - l)$ and hence for $0 \leq j \leq k < N$

$$\tilde{f}_0(j, k) = \frac{1}{2}(j(N - j) + k(N - k) - (k - j)(N - (k - j))) = j(N - k).$$

Since $\mathcal{L}_0 = \{(0, k) \mid 0 \leq k < N\}$ this implies in particular

$$\begin{cases} \tilde{f}_0(L) = 0 & \text{if } L \in \mathcal{L}_0 \\ \tilde{f}_0(L) > 0 & \text{if } L \notin \mathcal{L}_0 \end{cases} \quad (3.3)$$

and by the regularity expressed in Lemma 2.4 (here we need only the base case in its inductive proof) we conclude that $H_{\mathcal{L}_0} = \{f_0\}$.

The (in)equalities (3.3) show that $f_0 \in \Delta(X)$, and to conclude that $f_0 \in E(X)$ just take $y = 0$ in (3.2). \square

Theorem 3.6 *Consider the N -point metric space $X = \{0, 1, \dots, N - 1\}$ with metric $d(j, k) = |k - j|(N - |k - j|)$. Then the 0-faces of its injective hull $E(X)$, realized as in Definition 3.2, are precisely*

$$H_{\mathcal{L}_\lambda} = \bigcap_{L \in \mathcal{L}_\lambda} \{f \in \mathbb{R}^X \mid \tilde{f}(L) = 0\} =: \{f_\lambda\}$$

where λ runs through the partitions in \mathbb{Y}_N , and \mathcal{L}_λ is the outer rim of λ (as defined in Proposition 2.3). The solutions $f_\lambda \in \mathbb{R}^X$ have more explicit descriptions:

1. f_λ is determined by the following recursive procedure: $f_0(l) = l(N - l)$; and if λ covers μ such that $\mathcal{L}_\mu \ni (j, k + 1) \notin \mathcal{L}_\lambda$ for some pair (j, k) with $0 \leq j < k \leq N - 1$, then

$$f_\lambda(l) = f_\mu(l) + \begin{cases} 1 & \text{for } l = 0, \dots, j, \\ -1 & \text{for } l = j + 1, \dots, k, \\ 1 & \text{for } l = k + 1, \dots, N - 1. \end{cases} \quad (3.4)$$

2. f_λ can be described directly via $f_\lambda(0) = |\lambda|$ and by using the cyclic action. For $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{Y}_N$ let $\lambda_{m+1} = \dots = \lambda_{N-\lambda_1} = 0$ (for the empty partition put $m = 1$, $\lambda_1 = 0$). Then

$$\sigma(\lambda) = (\lambda_2 + 1, \dots, \lambda_{N-\lambda_1} + 1)$$

is the cyclic action alluded to in Remark 2.5. Its inverse τ (note the intended coincidence with the common notation for an Auslander-Reiten translation) is then $\tau(\lambda) = (\sigma(\lambda'))'$ and explicitly

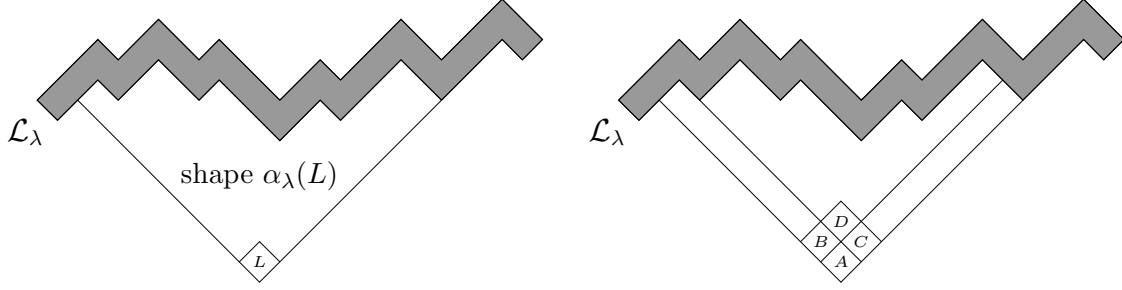
$$\tau(\lambda) = (N - m - 1, \lambda_1 - 1, \dots, \lambda_m - 1) \big|_{\text{remove trailing zeros}}$$

(for the empty partition put $m = 0$, $\lambda_1 = 1$). The formula for f_λ is

$$f_\lambda(l) = |\tau^l(\lambda)|. \quad (3.5)$$

Proof. Let $\lambda \in \mathbb{Y}_N$. By Lemma 2.4 the N affine hyperplanes $(H_L)_{L \in \mathcal{L}_\lambda}$ (Definition 3.4) intersect in a single point f_λ .

For a site $L \in \mathfrak{X} - \mathcal{L}_\lambda$ consider the partition $\alpha_\lambda(L)$ which is defined as follows: its Young diagram has the shape that is bordered by \mathcal{L}_λ and whose “bottom box” is at the position of the site L . For $L \in \mathcal{L}_\lambda$ let $\alpha_\lambda(L)$ be the empty partition.



In particular, we recover λ as

$$\lambda = \alpha_\lambda(L)|_{L=(0,0)=(0,N)}. \quad (3.6)$$

For a quadruple (A, B, C, D) of sites as depicted on the right side in the illustration above we get of course by considering the areas

$$|\alpha_\lambda(A)| + |\alpha_\lambda(D)| - |\alpha_\lambda(B)| - |\alpha_\lambda(C)| = 1 \quad (3.7)$$

so that finally

$$\tilde{f}_\lambda(L) = |\alpha_\lambda(L)|. \quad (3.8)$$

In fact, for $L \in \mathcal{L}_\lambda$ both sides in (3.8) vanish. The formula (3.7) computes the values $|\alpha_\lambda(L)|$ for all $L \in \mathfrak{X}$ recursively from the values $|\alpha_\lambda(L)| = 0$ for $L \in \mathcal{L}_\lambda$. On the other hand, by Lemma 3.3, $\tilde{f}_\lambda(L)$ satisfies the same recursive formula. Hence (3.8) holds true.

Remark 3.7 From (3.6) and (3.8) we get the formula (3.5) for the case $l = 0$, namely, $f_\lambda(0) = |\lambda|$. More generally, if $\lambda = (\lambda_1, \dots, \lambda_m)$ with dual partition $\lambda' = (\lambda'_1, \dots, \lambda'_{m'})$, then

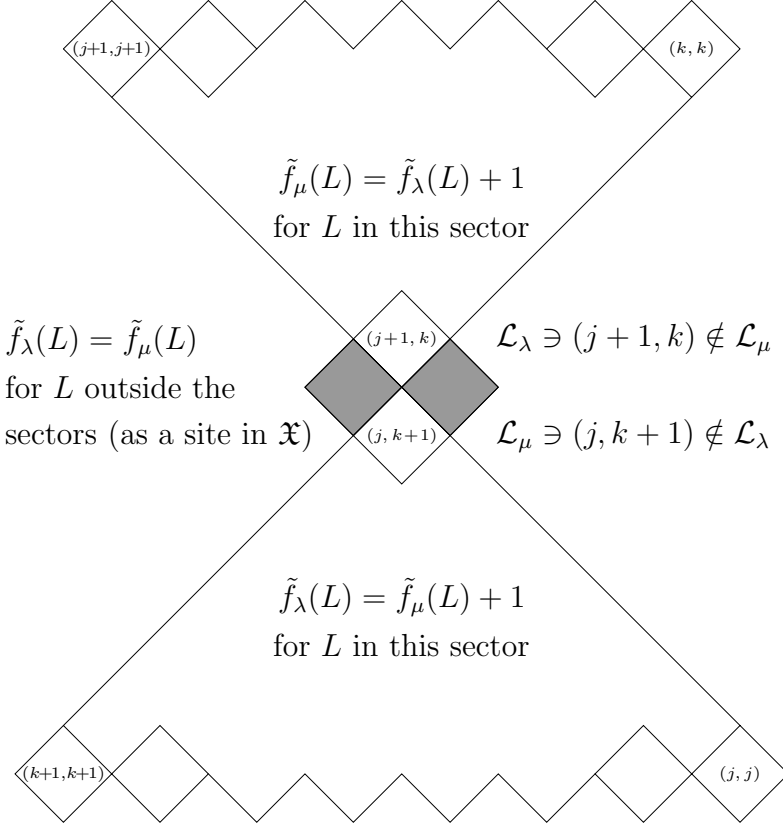
$$\begin{aligned} \tilde{f}_\lambda(0, k) &= \lambda'_{k+1} + \dots + \lambda'_{m'} & (0 \leq k \leq m'), \\ \tilde{f}_\lambda(0, N - k) &= \lambda_{k+1} + \dots + \lambda_m & (0 \leq k \leq m). \end{aligned}$$

The formula (3.5) follows from $\alpha_{\tau(\lambda)}(L) = \alpha_\lambda(\sigma_{\mathfrak{X}}(L))$, where $\sigma_{\mathfrak{X}} : \mathfrak{X} \rightarrow \mathfrak{X}$ is the translation $(j, k) \mapsto (j + 1, k + 1)$ in the Möbius strip, together with (3.6) and (3.8).

Before we continue with the first part of Theorem 3.6, we use the geometric interpretation to compare \tilde{f}_λ and \tilde{f}_μ if λ covers μ . Their outer rims \mathcal{L}_λ and \mathcal{L}_μ satisfy

$$\mathcal{L}_\lambda = (\mathcal{L}_\mu - \{(j, k + 1)\}) \cup \{(j + 1, k)\}.$$

The following picture illustrates the situation.

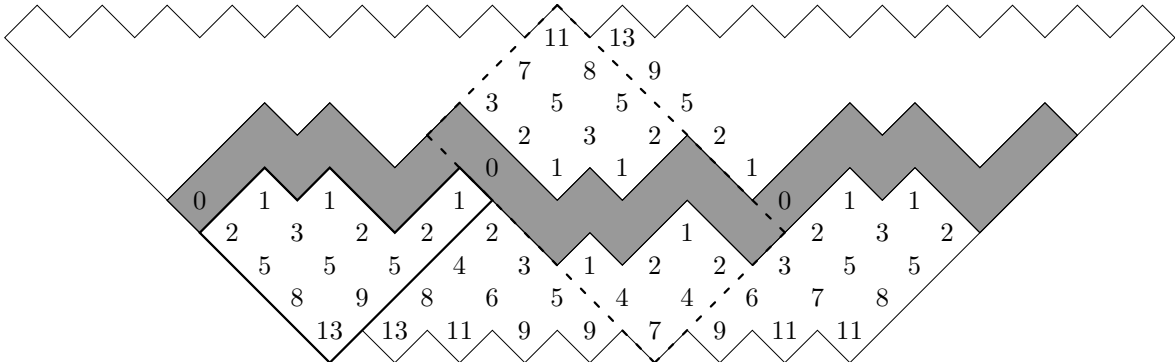


Hence $\tilde{f}_\lambda(L) - \tilde{f}_\mu(L) \in \{1, -1, 0\}$ depending on the position of the site L . In particular, this proves the formula (3.4) for the boundary values.

Let us continue with the proof of the first part of Theorem 3.6. From (3.8) we have $f_\lambda \in \Delta(X)$. Remark 2.2 say that for each $j \in X$ there exists $k \in X$ such that the site $L := (j, k) \in \mathcal{L}_\lambda \subseteq \mathfrak{X}$, and hence $\tilde{f}_\lambda(L) = 0$, which shows that $f_\lambda \in \mathbf{E}(X)$.

It is clear that $\mathbb{Y}_N \ni \lambda \mapsto f_\lambda \in \mathbb{R}^X$ is an injective function, for instance because $\mathcal{L}_\lambda = \{L \in \mathfrak{X} \mid \tilde{f}_\lambda(L) = 0\}$ determines λ as in (3.6). Hence we exhibited $|\mathbb{Y}_N| = 2^{N-1}$ 0-faces in $\mathbf{E}(X)$, which is the maximal possible number of 0-faces in the injective hull of a metric space with N points (the $v = 0$ case of the following Theorem 3.9 by Herrmann and Joswig). Hence the 0-faces of $\mathbf{E}(X)$ are precisely $(\{f_\lambda\})_{\lambda \in \mathbb{Y}_N}$. \square

Example Here is an example that illustrates the formula (3.8) with $N = 9$ for the partition $\lambda = (5, 3, 3, 2)$. Each site in \mathfrak{X} is represented thrice in the picture (in other words, it shows three fundamental domains in the universal cover). The displayed numbers are the values of $\tilde{f}_\lambda(L)$. (The dashed rectangle will serve as an illustration of Remark 3.8.)



Remark 3.8 For $(j, k) \in \mathcal{L}_\lambda$ (with $0 \leq j \leq k < N$) the formula

$$f_\lambda(j) + f_\lambda(k) = d(j, k) \quad (3.9)$$

from Theorem 3.6 (see also Definition 3.4) has the following interpretation: The shapes of $\alpha_\lambda(L)|_{L=(j,j)=(j,j+N)}$ and $\alpha_\lambda(L)|_{L=(k,k)}$ together with the rim \mathcal{L}_λ emanating from the position (j, k) and ending at the position $(k, j + N)$ ($= (j, k)$ as a site in \mathfrak{X}) tessellate the $(N - (k - j) + 1) \times (k - j + 1)$ rectangle with corners (j, k) , $(j, j + N)$, $(k, j + N)$, (k, k) . Hence

$$\begin{aligned} f_\lambda(j) + f_\lambda(k) &= |\alpha_\lambda(L)|_{L=(j,j)=(j,j+N)}| + |\alpha_\lambda(L)|_{L=(k,k)}| \\ &= (N - (k - j) + 1)(k - j + 1) - (N + 1) = d(j, k). \end{aligned}$$

As an illustration we look at the example above and take $(j, k) = (5, 8)$. The picture shows the corresponding dashed 7×4 rectangle. So

$$f_\lambda(5) + f_\lambda(8) = 7 + 11 = 7 \cdot 4 - 10.$$

The formula (4.4) in Section 4 is a continuous analogue of (3.9).

Theorem 3.9 (S. Herrmann, M. Joswig [HJ]) *The number of v -faces in an injective hull of a metric space with N points is at most*

$$2^{N-2v-1} \frac{N}{N-v} \binom{N-v}{v},$$

and for each N there is a metric space attaining those upper bounds uniformly for all v .

Let $\lambda \in \mathbb{Y}_N$ with outer rim \mathcal{L}_λ as in Proposition 2.3. Let us enumerate those sites $L_1 = (j_1 + 1, k_1), \dots, L_s = (j_s + 1, k_s)$ in the outer rim \mathcal{L}_λ that correspond to the inner corners $(j_1, k_1 + 1), \dots, (j_s, k_s + 1)$ of λ . The partition λ covers exactly s partitions, namely those partitions μ that are got by removing from λ one of its inner corners. The 2^s partitions ν that are got from λ by removing i (running from 0 to s) of its inner corners constitute a Boolean lattice. We shall recognize that the convex hull of those 2^s vertices f_ν belongs to the injective hull $E(X)$. In fact, let $f = \sum_\nu a_\nu f_\nu$ be such a convex combination. Then from $\tilde{f}_\nu(L) \geq 0$ for all $L \in \mathfrak{X}$ we have $\tilde{f}(L) \geq 0$ for all $L \in \mathfrak{X}$, that is, $f \in \Delta(X)$. For each $L \in \mathcal{L}_\lambda - \{L_1, \dots, L_s\}$ we have $\tilde{f}_\nu(L) = 0$ and hence $\tilde{f}(L) = 0$. Since $(j_i, k_i), (j_i + 1, k_i + 1) \in \mathcal{L}_\lambda - \{L_1, \dots, L_s\}$, we still get (as a generalization of Remark 2.2)

$$X = \bigcup_{(j,k) \in \mathcal{L}_\lambda - \{L_1, \dots, L_s\}} \{j, k\} \quad (3.10)$$

from which we conclude that $f \in E(X)$. (Note also that from (3.10) we get $s \leq \lfloor \frac{N}{2} \rfloor$. This is Develin's bound for the maximal possible dimension of the injective hull of a metric space with N points (see [De]), a result that is superseded by Theorem 3.9.)

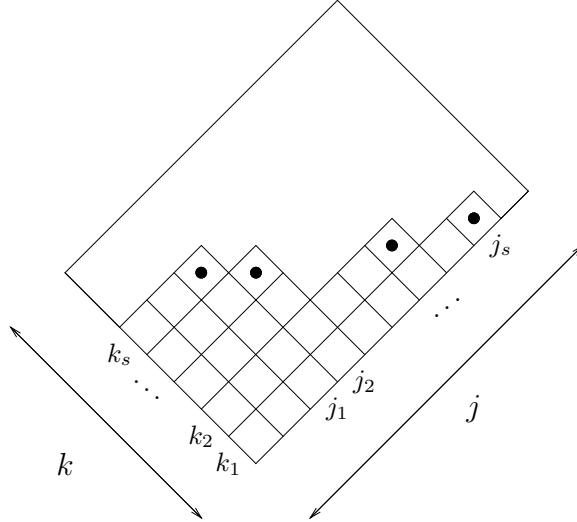
A systematic count of v -faces can be done as follows: Each choice of v among the s inner corners of λ results in a v -face with f_λ as its "top vertex". In this way, we get $\binom{s}{v}$ v -faces with "top vertex" f_λ . The total number of v -faces that we get from this procedure (and we shall soon see that every v -face occurs in such a way) is thus

$$\#(v\text{-faces in } E(X)) = \sum_{\lambda \in \mathbb{Y}_N} \binom{s(\lambda)}{v} \quad (3.11)$$

where $s(\lambda)$ denotes the number of inner corners of the partition λ .

Lemma 3.10 *The number of partitions with first part at most j and having at most k parts that cover exactly s partitions is $a(j, k, s) := \binom{j}{s} \binom{k}{s}$.*

Proof. We have to count the partitions with first part at most j and at most k parts with exactly s inner corners. This amounts to choosing s integers $1 \leq j_1 < \dots < j_s \leq j$ and s integers $1 \leq k_1 < \dots < k_s \leq k$ because the pairs (j_i, k_{s+1-i}) for $1 \leq i \leq s$ determine the positions of the s inner corners of a partition in the given range in an evident way as illustrated in the picture.

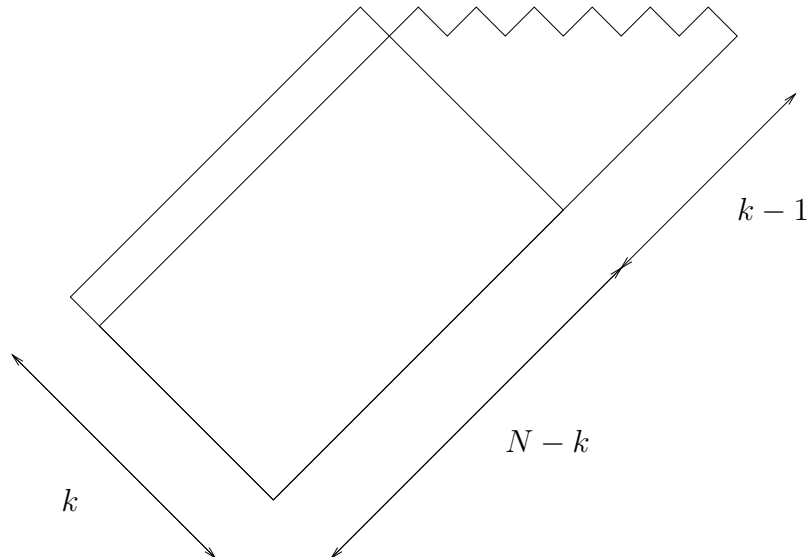


The correspondence is clearly bijective. □

Proposition 3.11 *The number of partitions in \mathbb{Y}_N that cover exactly s partitions (that is, with exactly s inner corners) is $\binom{N}{2s}$.*

Proof. Let us compute the number in question by inclusion-exclusion and using the previous lemma and its notation $a(j, k, s) = \binom{j}{s} \binom{k}{s}$ for the number of partitions with first part at most j and having at most k parts that cover exactly s partitions.

The formula is obviously true for $s = 0$ (corresponding to the empty partition, which is contained in every \mathbb{Y}_N); hence we assume that $s \geq 1$. The following figure explains the inductive step for the inclusion-exclusion argument.



The number $A(N, k, s)$ of partitions in \mathbb{Y}_N having at most k parts that cover exactly s partitions is

$$A(N, k, s) = A(N, k-1, s) + a(N-k, k, s) - a(N-k, k-1, s).$$

Furthermore, $A(N, 1, s) = a(N-1, 1, s)$. The number $A(N, N-1, s)$ that we want to compute is therefore

$$\begin{aligned} A(N, N-1, s) &= \sum_{k=1}^{N-1} a(N-k, k, s) - \sum_{k=2}^{N-1} a(N-k, k-1, s) \\ &= \sum_{k=1}^{N-1} \binom{N-k}{s} \binom{k}{s} - \sum_{k=2}^{N-1} \binom{N-k}{s} \binom{k-1}{s} \end{aligned}$$

and combining the binomial coefficients $\binom{k}{s} - \binom{k-1}{s} = \binom{k-1}{s-1}$ after adding the zero summand $-\binom{N-k}{s} \binom{k-1}{s} \Big|_{k=1}$ (recall that $s \geq 1$) in the second sum, we get

$$= \sum_{k=1}^{N-1} \binom{N-k}{s} \binom{k-1}{s-1} = \binom{N}{2s},$$

where the last equality is clear from the following combinatorial interpretation: To choose $2s$ integers $1 \leq i_1 < \dots < i_{s-1} < i_s < i_{s+1} < \dots < i_{2s} \leq N$ is tantamount to first fixing $i_s = k \in \{1, \dots, N-1\}$ and then choosing $s-1$ integers in $\{1, \dots, k-1\}$ and choosing s integers in $\{k+1, \dots, N\}$. \square

Lemma 3.12 *The number of v -faces in $E(X_N)$ is (at least and in fact exactly)*

$$\#(v\text{-faces in } E(X_N)) = \sum_{s=v}^{\lfloor N/2 \rfloor} \binom{N}{2s} \binom{s}{v}.$$

Proof. This is clear from Proposition 3.11 together with (3.11). \square

Theorem 3.13 *Consider the N -point metric space $X_N = \{0, 1, \dots, N-1\}$ with metric $d(j, k) = |k-j|(N-|k-j|)$. The number of v -faces in its injective hull $E(X_N)$ (as in Definition 3.2) is*

$$\#(v\text{-faces in } E(X_N)) = 2^{N-2v-1} \frac{N}{N-v} \binom{N-v}{v}. \quad (3.12)$$

Proof. For $v = 0$ we have already proved this in Theorem 3.6 because $|\mathbb{Y}_N| = 2^{N-1}$. Hence we assume that $v \geq 1$. Using Lemma 3.12 and three instances of the binomial series expansion (as a formal power series)

$$\sum_{n=a}^{\infty} \binom{n}{a} z^n = \frac{z^a}{(1-z)^{a+1}} \quad \text{for } a \in \mathbb{Z}_{\geq 0} \quad (*)$$

(namely, twice in the direction from left to right and once in the opposite direction), we

compute the generating series

$$\begin{aligned}
& \sum_{N \geq 2} \#(v\text{-faces in } E(X_N)) q^N \\
&= \sum_{N \geq 2} \sum_{s=v}^{\lfloor N/2 \rfloor} \binom{N}{2s} \binom{s}{v} q^N = \sum_{s=v}^{\infty} \sum_{N=2s}^{\infty} \binom{N}{2s} \binom{s}{v} q^N = \sum_{s=v}^{\infty} \binom{s}{v} \sum_{N=2s}^{\infty} \binom{N}{2s} q^N \\
&\stackrel{(*)}{=} \sum_{s=v}^{\infty} \binom{s}{v} \frac{q^{2s}}{(1-q)^{2s+1}} = \frac{1}{(1-q)} \sum_{s=v}^{\infty} \binom{s}{v} \left(\frac{q^2}{(1-q)^2} \right)^s \\
&\stackrel{(*)}{=} \frac{1}{(1-q)} \frac{\left(\frac{q^2}{(1-q)^2} \right)^v}{\left(1 - \frac{q^2}{(1-q)^2} \right)^{v+1}} = \frac{(1-q)q^{2v}}{(1-2q)^{v+1}} = (1-q)q^v 2^{-v} \frac{(2q)^v}{(1-2q)^{v+1}} \\
&\stackrel{(*)}{=} (1-q)q^v 2^{-v} \sum_{n=v}^{\infty} \binom{n}{v} (2q)^n \\
&= q^v 2^{-v-1} \sum_{N=2v}^{\infty} 2 \binom{N-v}{v} (2q)^{N-v} - q^{v+1} 2^{-v} \sum_{N=2v+1}^{\infty} \binom{N-v-1}{v} (2q)^{N-v-1}
\end{aligned}$$

and using $2 \binom{N-v}{v} - \binom{N-v-1}{v} = \frac{N}{N-v} \binom{N-v}{v}$ and $\binom{N-v-1}{v} \big|_{N=2v} = 0$, we get finally

$$= \sum_{N=2v}^{\infty} 2^{N-2v-1} \frac{N}{N-v} \binom{N-v}{v} q^N.$$

Hence the formula (3.12) is proved. According to Theorem 3.9, X_N is a metric space such that $E(X_N)$ attains the maximal possible number of v -faces for all v . \square

Remark 3.14 Theorem 1.3 from the introductory section follows as a corollary. In fact, the vertices of $E(X_N)$ were described in Theorem 3.6 as $f_\lambda \in E(X_N)$ for $\lambda \in \mathbb{Y}_N$. The considerations after Theorem 3.9 showed that if $\mathbb{Y}_N \ni \lambda$ covers μ , then the segment with endpoints f_λ and f_μ is an edge in $E(X_N)$. In this way we get $2^{N-3}N$ edges in $E(X_N)$, and according to Theorem 3.13, those $2^{N-3}N$ edges are all the 1-faces in $E(X_N)$.

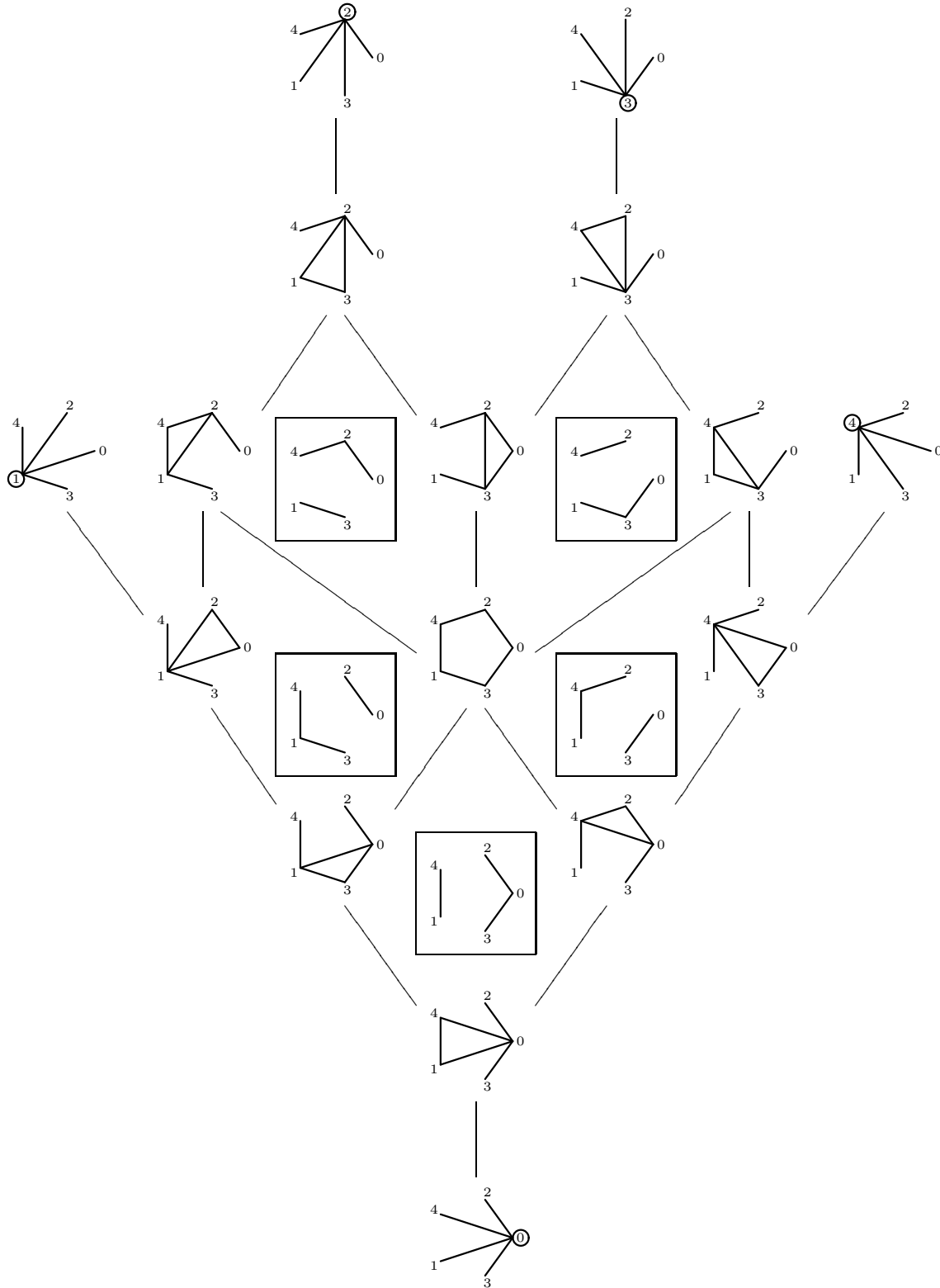
Remark 3.15 For $v = 1$ we get $2^{N-3}N$ edges in $E(X_N)$ and thus in $\text{Hasse}(\mathbb{Y}_N)$. This result generalizes to the fact that the Hasse diagram of the poset of abelian ideals in a Borel subalgebra of a complex simple Lie algebra of rank n has $2^{n-2}(n+1)$ edges (see [Pa, Theorem 4.1]).

For $v \geq 2$ such a simple (only rank-dependent) census breaks down. Recall that $E(X_5)$ (type A_4) has five 2-faces and no higher-dimensional faces. But the poset of abelian ideals in a Borel subalgebra of $\mathfrak{so}(8, \mathbb{C})$ (type D_4) contains a Boolean subposet of rank 3: If $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the simple roots and $\theta = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$ is the highest root, then the eight ideals in question are

$$\mathfrak{g}_\theta \oplus \mathfrak{g}_{\theta-\alpha_2} \oplus \begin{pmatrix} \mathfrak{g}_{\theta-\alpha_2-\alpha_1} \\ \text{or} \\ 0 \end{pmatrix} \oplus \begin{pmatrix} \mathfrak{g}_{\theta-\alpha_2-\alpha_3} \\ \text{or} \\ 0 \end{pmatrix} \oplus \begin{pmatrix} \mathfrak{g}_{\theta-\alpha_2-\alpha_4} \\ \text{or} \\ 0 \end{pmatrix},$$

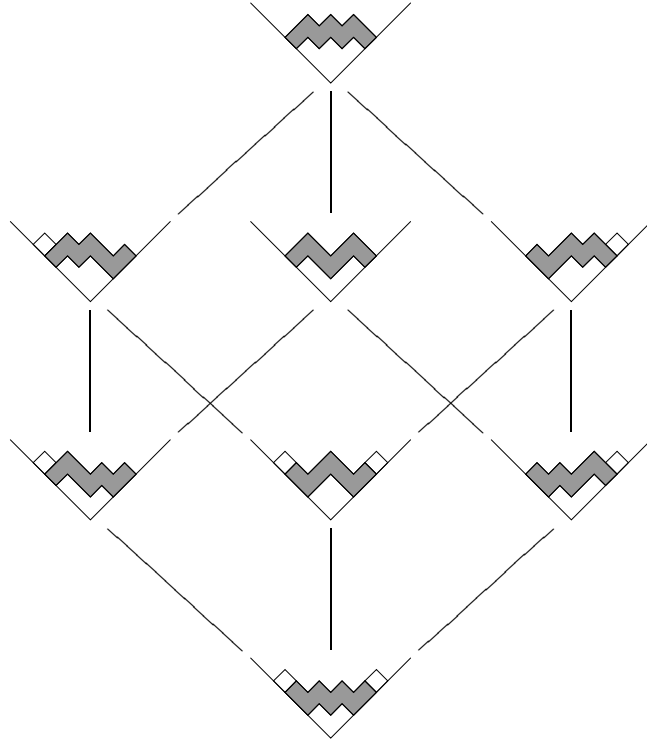
where \mathfrak{g}_φ denotes the root space for the root φ . Hence the Hasse diagram contains the 1-skeleton of a cube, which has one 3-face and six 2-faces.

Example (The Hasse diagram for $N = 5$) Its vertices $\lambda \in \mathbb{Y}_5$ are encoded by the graphs with vertex set X_5 , and there is an edge between j and k if and only if $(j, k) \in \mathcal{L}_\lambda$ (the outer rim of λ as defined in Proposition 2.3).

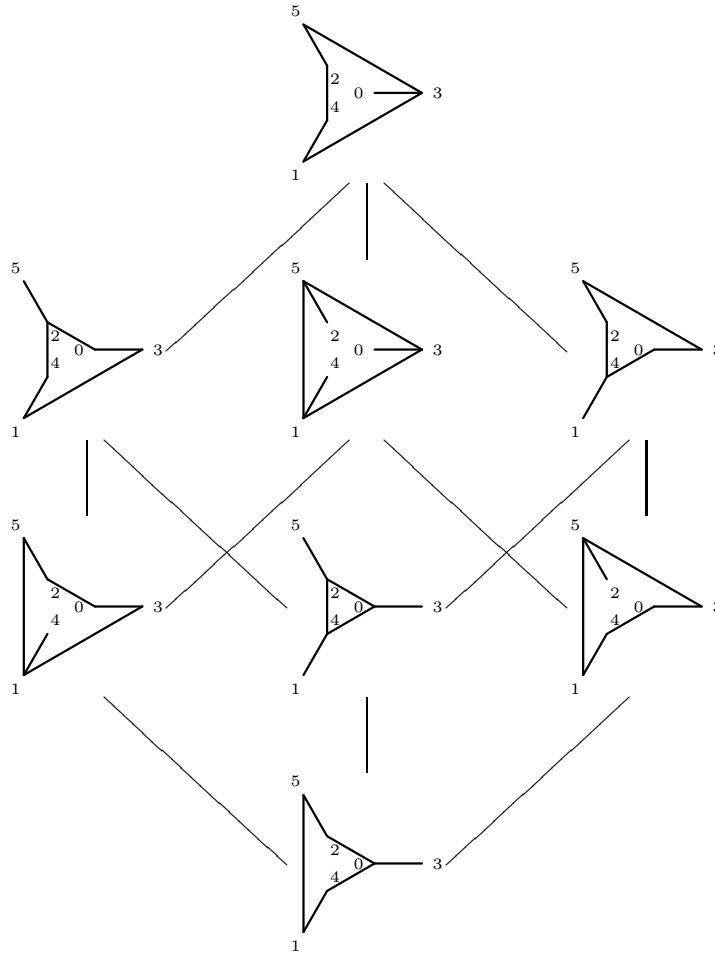


The five graphs in the framed boxes encode the 2-faces of $E(X_5)$ in an evident manner (and the twenty edges of \mathbb{Y}_5 or 1-faces of $E(X_5)$ are encoded by the obvious (but not separately displayed) spanning trees).

Example (The central cube for $N = 6$) The vertices in the first visualization show the outer rims (as defined in Proposition 2.3).



The second visualization shows the graphs on $\{0, 1, 2, 3, 4, 5\}$ that encode the outer rims.



Projection of the 1-skeleton to the plane

The points $j = 0, \dots, N-1 \in X$ correspond to the rectangular partitions $R_j = (j^{N-j})$, where (0^N) is synonymous with the empty partition. Recall that

$$f_{R_j}(k) = |k - j|(N - |k - j|). \quad (3.13)$$

The $N \times N$ symmetric circulant matrix $C_N := (f_{R_j}(k))_{j,k \in X}$ is invertible (for $N \geq 2$); in fact, its determinant is

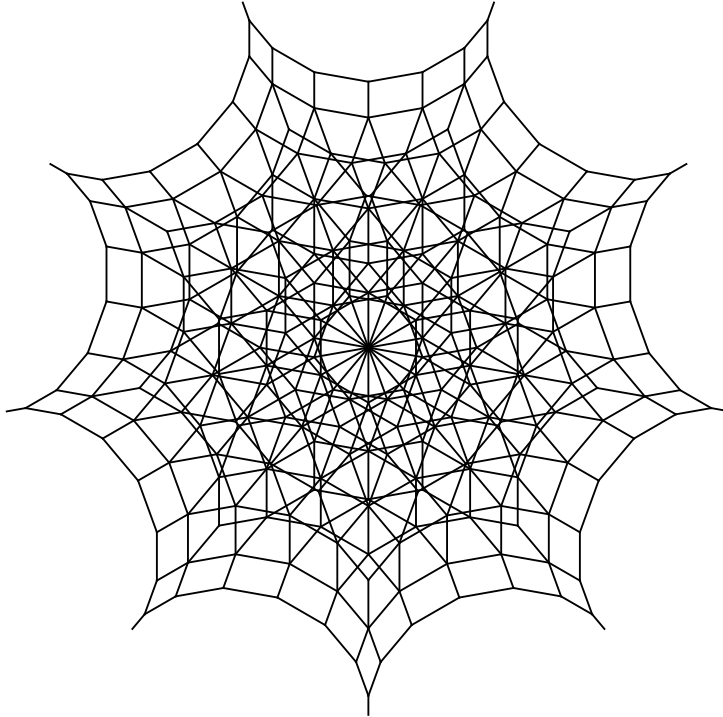
$$\begin{aligned} \det C_N &= \prod_{j=0}^{N-1} \left(\sum_{k=0}^{N-1} k(N-k) \exp\left(\frac{2\pi i j k}{N}\right) \right) \\ &= \frac{N(N^2-1)}{6} \prod_{j=1}^{N-1} \frac{2N \exp\left(\frac{2\pi i j}{N}\right)}{\left(1 - \exp\left(\frac{2\pi i j}{N}\right)\right)^2} = (-1)^{N+1} \frac{(2N)^{N-2} (N^2-1)}{3}, \end{aligned}$$

where the first equality follows by writing $\det C_N$ as the product of the eigenvalues of C_N (by using the obvious eigenvectors) and the rest is computation (given without details).

We use the matrix

$$P := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \cos \frac{2\pi}{N} & \dots & \cos \frac{2\pi j}{N} & \dots & \cos \frac{2\pi(N-1)}{N} \\ 0 & \sin \frac{2\pi}{N} & \dots & \sin \frac{2\pi j}{N} & \dots & \sin \frac{2\pi(N-1)}{N} \end{pmatrix} C_N^{-1}$$

to project \mathbb{R}^X to the plane, so that $Pf_{R_0}, \dots, Pf_{R_{N-1}}$ are the vertices of a regular N -gon. The following picture shows the image of the 1-skeleton of $E(X)$ for $N = 9$.



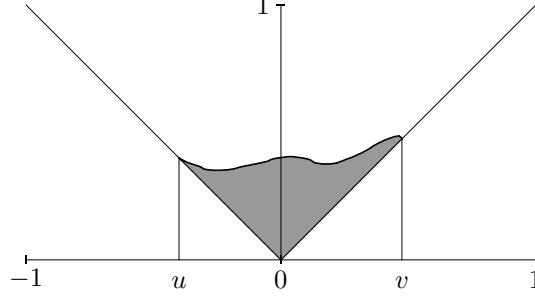
We have $Pf_\lambda = (0, 0)$ for the fixed point $\lambda = (4, 3, 2, 1)$ under the action of the cyclic group generated by τ (see Theorem 3.6), but also for $\lambda \in \{(3, 3, 3), (5, 2, 2, 2), (4, 4, 1, 1, 1)\}$, which is an orbit of size three, we have $Pf_\lambda = (0, 0)$. All the other orbits have size nine.

4 A continuous version

Let us consider as a continuous version of integer partitions certain functions

$$\begin{aligned} \Lambda : [u, v] &\longrightarrow \mathbb{R}_{\geq 0} \\ u &\longmapsto -u \\ v &\longmapsto v \end{aligned} \tag{4.1}$$

with $-1 \leq u \leq 0 \leq v \leq 1$ such that the set $\{(t, s) \mid t \in [u, v], |t| \leq s \leq \Lambda(t)\} \subseteq \mathbb{R}^2$



has an evident meaning of “continuous Young diagram” of the “continuous partition” Λ . That the parts of an integer partition are ordered decreasingly translates into the requirement that Λ must be 1-Lipschitz. We further require that Λ be defined on an interval $[u, v]$ of length at most 1 (which replaces the bound on the hook lengths for integer partitions in \mathbb{Y}_N). There is then a unique extension of Λ to a 1-Lipschitz function $\Lambda : [u, u+1] \rightarrow \mathbb{R}_{\geq 0}$ with $\Lambda(u+1) = u+1$, namely, $\Lambda(t) = t$ for $t \in [v, u+1]$. Note that $\Lambda(t) \leq 1$ for all $t \in [u, u+1]$ because for $u < t < u+1$ the 1-Lipschitz condition gives

$$\frac{\Lambda(t) + u}{t - u} = \frac{\Lambda(t) - \Lambda(u)}{t - u} \leq 1 \quad \text{and} \quad \frac{\Lambda(t) - u - 1}{u + 1 - t} = \frac{\Lambda(t) - \Lambda(u+1)}{u + 1 - t} \leq 1$$

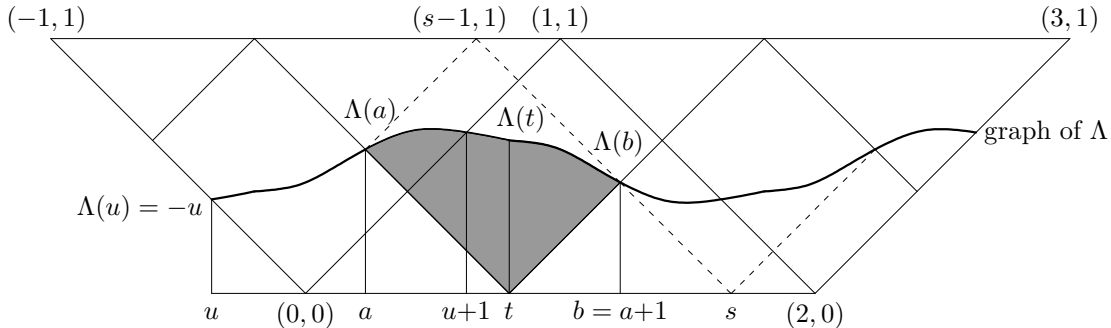
and hence

$$2\Lambda(t) - 1 = (\Lambda(t) + u) + (\Lambda(t) - u - 1) \leq (t - u) + (u + 1 - t) = 1.$$

The shifted antiperiodic extension defined by $\Lambda(t+1) = 1 - \Lambda(t)$ finally extends Λ to the real line $\Lambda : \mathbb{R} \rightarrow [0, 1]$, so that the graph of Λ becomes the preimage of a loop in the Möbius strip under the universal covering projection. We define

$$\mathbb{Y}_\infty := \{\Lambda : \mathbb{R} \rightarrow [0, 1] \mid \Lambda \text{ is 1-Lipschitz and } \Lambda(t+1) = 1 - \Lambda(t) \text{ for all } t\}.$$

For each $\Lambda \in \mathbb{Y}_\infty$ its graph intersects the line segment with endpoints $(-1, 1)$ and $(0, 0)$ in at least one (and generically exactly one) point. If $(u, -u)$ is a point in the intersection, then the restriction $\Lambda|_{[u, u+1]}$ defines a “continuous partition” as in (4.1) with $v = u+1$.



In analogy with the assignment $\mathbb{Y}_N \ni \lambda \mapsto f_\lambda$ we define $\mathbb{Y}_\infty \ni \Lambda \mapsto F_\Lambda$ by the (2-periodic) area function

$$F_\Lambda(t) = \int_a^b \Lambda(s) ds - \frac{1}{2}(\Lambda(a)^2 + \Lambda(b)^2)$$

where $b = a + 1$ and $t = a + \Lambda(a)$ and hence $t = b - \Lambda(b)$; moreover, $s = b + \Lambda(b)$.

For $u \leq a \leq u + 1$ and hence $0 \leq a + \Lambda(a) \leq 2 + 2u \leq a + 1 + \Lambda(a + 1) \leq 2$ we have

$$\begin{aligned} F_\Lambda(t) &= F_\Lambda(a + \Lambda(a)) = \int_a^{u+1} \Lambda(s) ds + \int_{u+1}^{a+1} \Lambda(s) ds - \frac{1}{2}(\Lambda(a)^2 + \Lambda(a+1)^2) \\ &= \int_a^{u+1} \Lambda(s) ds + \int_u^a (1 - \Lambda(s)) ds - \frac{1}{2}(\Lambda(a)^2 + (1 - \Lambda(a))^2) \\ &= a - u - \int_u^a \Lambda(s) ds + \int_a^{u+1} \Lambda(s) ds + \Lambda(a)(1 - \Lambda(a)) - \frac{1}{2}, \end{aligned} \quad (4.2)$$

$$\begin{aligned} F_\Lambda(s) &= F_\Lambda(b + \Lambda(b)) = F_\Lambda(a + 1 + \Lambda(a + 1)) = F_\Lambda(2 + a - \Lambda(a)) \\ &= \int_{a+1}^{u+2} \Lambda(s) ds + \int_{u+2}^{a+2} \Lambda(s) ds - \frac{1}{2}(\Lambda(a+1)^2 + \Lambda(a+2)^2) \\ &= \int_a^{u+1} (1 - \Lambda(s)) ds + \int_u^a \Lambda(s) ds - \frac{1}{2}((1 - \Lambda(a))^2 + \Lambda(a)^2) \\ &= u + 1 - a + \int_u^a \Lambda(s) ds - \int_a^{u+1} \Lambda(s) ds + \Lambda(a)(1 - \Lambda(a)) - \frac{1}{2}. \end{aligned} \quad (4.3)$$

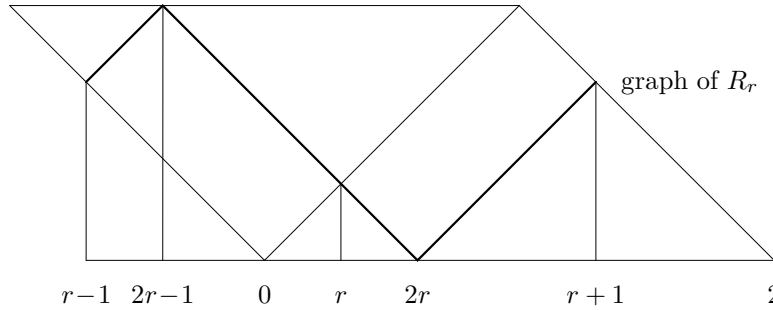
The equality

$$F_\Lambda(a + \Lambda(a)) + F_\Lambda(b + \Lambda(b)) = 2\Lambda(a)\Lambda(b) \quad (\text{for } b = a + 1) \quad (4.4)$$

follows directly from the previous picture by considering the rectangle with corners at $(a, \Lambda(a))$, $(t, 0)$, $(b, \Lambda(b))$, $(s - 1, 1)$; alternatively, (4.4) follows by adding (4.2) and (4.3).

For $r \in [0, 1]$ let $R_r \in \mathbb{Y}_\infty$ be the function that satisfies

$$R_r(t) = \begin{cases} t + 2 - 2r & \text{if } r - 1 \leq t \leq 2r - 1, \\ -t + 2r & \text{if } 2r - 1 \leq t \leq r. \end{cases}$$



Note that $R_0 = R_1$. Note also that R_r is the unique function in \mathbb{Y}_∞ that vanishes at $2r$. The 2-periodic function F_{R_r} is then

$$F_{R_r}(t) = \frac{1}{2}(t - 2r)(2r + 2 - t) \quad \text{for } 2r \leq t \leq 2r + 2.$$

We define the metric space X_∞ as the subspace

$$X_\infty := \{R_r \mid r \in [0, 1]\} \subseteq \mathbb{Y}_\infty.$$

Two alternative characterizations are

$$X_\infty = \{\Lambda \in \mathbb{Y}_\infty \mid \exists t \in \mathbb{R} : \Lambda(t) = 0\} = \{\Lambda \in \mathbb{Y}_\infty \mid \exists t \in \mathbb{R} : F_\Lambda(t) = 0\}.$$

The distance function is

$$D(r, s) := d(R_r, R_s) = \|F_{R_r} - F_{R_s}\|_\infty = 2|s - r|(1 - |s - r|).$$

For $r, s \in [0, 1]$ the equality

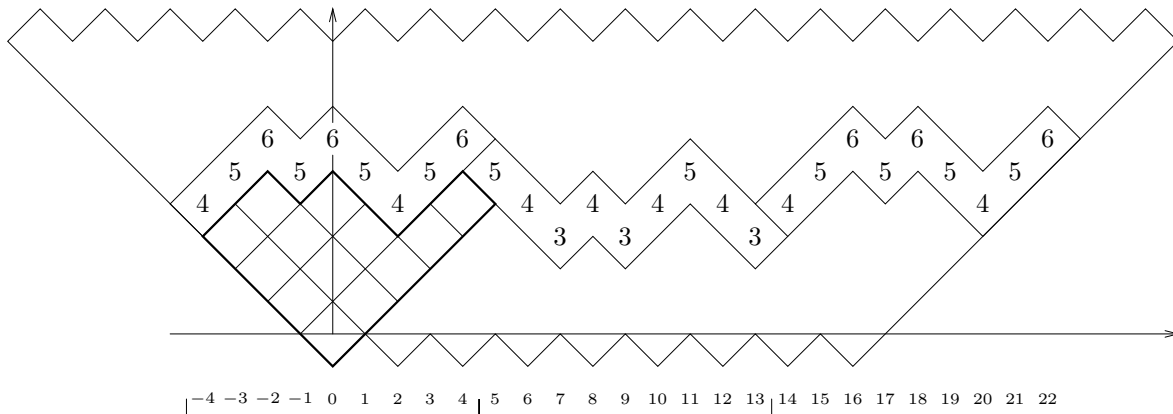
$$F_{R_r}(2s) = 2|s - r|(1 - |s - r|)$$

is an analogue of (3.13).

A short dictionary

$\mathbb{Y}_N \quad (N \in \mathbb{Z}_{\geq 2})$	\mathbb{Y}_∞
$\lambda \in \mathbb{Y}_N$	$\Lambda : [u, u + 1] \rightarrow [0, 1]$ 1-Lipschitz with $\Lambda(u) = -u$ and $\Lambda(u + 1) = u + 1$ extended by $\Lambda(t + 1) = 1 - \Lambda(t)$ to all $t \in \mathbb{R}$
$(0, 0), (0, N), (N, N); (j, k)$	$(-1, 1), (0, 0), (1, 1); (\frac{j}{N} + \frac{k}{N} - 1, \frac{j}{N} - \frac{k}{N} + 1)$
$(j, k) \in \mathcal{L}_\lambda$ (for $0 \leq j \leq k \leq N$)	$(a, \Lambda(a))$ (for $u \leq a \leq u + 1$)
$d(j, k) = k - j (N - k - j)$	$D(r, s) = 2 s - r (1 - s - r)$
$\forall (j, k) \in \mathcal{L}_\lambda :$ $f_\lambda(j) + f_\lambda(k) = d(j, k)$ $= d(0, k - j)$	for $b = a + 1$: $F_\Lambda(a + \Lambda(a)) + F_\Lambda(b + \Lambda(b)) = 2\Lambda(a)\Lambda(b)$ $= D(0, \Lambda(a)) = D(0, \Lambda(b))$
X_N	X_∞

Of course one could elaborate on the correspondence by first putting the representation of integer partitions into a form analogous to the representation of “continuous partitions”. For instance, look at the partition $\lambda = (5, 3, 3, 2) \in \mathbb{Y}_N$ with $N = 9$.



The function $\Lambda_9 : [-4, 4] \cap \mathbb{Z} \rightarrow [0, 9] \cap \mathbb{Z}$

t	-4	-3	-2	-1	0	1	2	3	4
$\Lambda_9(t)$	4	5	6	5	6	5	4	5	6

describes the heights of the box positions in the outer rim of λ . In general, for $\lambda \in \mathbb{Y}_N$ we have the corresponding function Λ_N defined on a set of N consecutive integers that contains 0, and we extend it to a function $\Lambda_N : \mathbb{Z} \rightarrow [0, N] \cap \mathbb{Z}$ by requiring the symmetry $\Lambda_N(t + N) = N - \Lambda_N(t)$. In the example we have $\Lambda_9(-4) = 4$ and $\Lambda_9(5) = 5$, which is reminiscent of the boundary condition $\Lambda(u) = -u$ and $\Lambda(u + 1) = u + 1$ for a “continuous partition” $\Lambda : [u, u + 1] \rightarrow [0, 1]$. Let us also write down the formula

$$\Lambda_N(0) = 2 \cdot \sqrt{\#(\text{boxes in the Durfee square of the partition } \lambda)}.$$

5 Some conclusions and outlook

The realization of $\text{Hasse}(\mathbb{Y}_N)$ as the 1-skeleton of the $\lfloor \frac{N}{2} \rfloor$ -dimensional polyhedral complex $E(X_N)$ embedded in \mathbb{R}^N (in fact, in its nonnegative orthant) that is stable under cyclic permutation of the coordinates, provides a new geometric realization of the N -fold cyclic symmetry of $\text{Hasse}(\mathbb{Y}_N)$.

When one looks at the graphs $\text{Hasse}(\mathbb{Y}_N)$ depicted in [Su1] for $N = 5, 6, 7, 8$, then the 1-skeleta of five squares, one cube, seven cubes, respectively one tesseract are eye-catching. To some extent, the construction of $E(X_N)$ gives a meaning to those $\lfloor \frac{N}{2} \rfloor$ -cubes. In general, Theorem 3.13 counts all those as well as the lower-dimensional v -cubes.

The works of C. Berg and M. Zabrocki [BZ] and H. Thomas and N. Williams [TW] generalize the cyclic symmetries and prove an instance of the cyclic sieving phenomenon in that framework. One may ask whether the metric geometry approach developed here for the original cyclic symmetries can be extended in some nice way to the generalized version.

Homotopically nontrivial loops (as specified in Definition 2.1) play an important role. An investigation of loops (self-avoiding or not, of given length and with a prescribed winding number) in various sorts of discrete strips (also with other than A_{N-1} type Dynkin diagram fibres) might involve interesting combinatorics.

Let us conclude with some geometry. The maximal coordinate sum of points in $E(X_N)$ is attained for the N points in the orbit of f_0 ,

$$\max\{\|f\|_1 \mid f \in E(X_N)\} = \|f_0\|_1 = \sum_{j=0}^{N-1} j(N-j) = \frac{1}{6}(N^3 - N).$$

For N odd there is a unique partition $\lambda \in \mathbb{Y}_N$ with

$$\min\{\|f\|_1 \mid f \in E(X_N)\} = \|f_\lambda\|_1 = \frac{1}{8}(N^3 - N),$$

namely the staircase partition $\lambda = (\frac{N-1}{2}, \frac{N-3}{2}, \dots, 1)$, such that $f_\lambda(l) = \frac{1}{8}(N^2 - 1)$ for all l . For N even there is a central $\frac{N}{2}$ -dimensional cube, whose points have minimal 1-norm

$$\min\{\|f\|_1 \mid f \in E(X_N)\} = \frac{1}{8}N^3.$$

The vertices of this $\frac{N}{2}$ -cube are f_ν where ν is one of the $2^{N/2}$ partitions that are got by removing from the staircase partition $(\frac{N}{2}, \frac{N-2}{2}, \dots, 1)$ any subset of its inner corners. The barycentre of this $\frac{N}{2}$ -cube has all coordinates $\frac{1}{8}N^2$, but no vertex of $\text{Hasse}(\mathbb{Y}_N)$ is fixed under the cyclic action (which is also a very particular case of the cyclic sieving phenomenon established in [TW]).

Recall that it is the supremum norm in \mathbb{R}^N that endows $E(X_N)$ with the metric that characterizes this space as an injective hull. But the polyhedral complex $E(X_N)$ can also be looked at from a more geometric point of view by using the Euclidean metric, so that all the edges have length \sqrt{N} by Theorem 3.6. In the Euclidean metric the ‘central cube’ for $N = 6$ is actually a rhombohedron with six congruent faces, namely, rhombi with diagonals of lengths $2\sqrt{2}$ and 4; one of the diagonals of the rhombohedron has length $\sqrt{6}$ and the other three have length $\sqrt{22}$ (which is $\sqrt{\frac{1}{12}(N^3 + 8N)}\Big|_{N=6}$).

References

- [BZ] C. Berg, M. Zabrocki, Symmetries on the lattice of k -bounded partitions, preprint (2011).
arXiv:1111.2783 [math.CO]
- [De] M. Develin, Dimensions of tight spans, *Ann. Comb.* **10** (2006) 53–61.
doi:10.1007/s00026-006-0273-y
- [Dr1] A. Dress, Trees, tight extensions of metric spaces, and the cohomological dimension of certain groups: a note on combinatorial properties of metric spaces, *Adv. Math.* **53** (1984) 321–402.
doi:10.1016/0001-8708(84)90029-X
- [Dr2] A. Dress, Towards a classification of transitive group actions on finite metric spaces, *Adv. Math.* **74** (1989) 163–189.
doi:10.1016/0001-8708(89)90008-X
- [HJ] S. Herrmann, M. Joswig, Bounds on the f -vectors of tight spans. *Contrib. Discrete Math.* **2** (2007) 161–184.
doi:10.5115/sy54m91r1
- [HKM] K. Huber, J. Koolen, V. Moulton, The tight span of an antipodal metric space: Part II—geometrical properties, *Discrete Comput. Geom.* **31** (2004) 567–586.
doi:10.1007/s00454-004-0777-3
- [Is] J. Isbell, Six theorems about injective metric spaces, *Comment. Math. Helv.* **39** (1964) 65–76.
doi:10.5169/seals-29876
- [Pa] D. Panyushev, The poset of positive roots and its relatives, *J. Algebr. Comb.* **23** (2006) 79–101.
doi:10.1007/s10801-006-6030-9
- [Su1] R. Suter, Young’s lattice and dihedral symmetries, *Eur. J. Comb.* **23** (2002) 233–238.
doi:10.1006/eujc.2001.0541
- [Su2] R. Suter, Abelian ideals in a Borel subalgebra of a complex simple Lie algebra, *Invent. Math.* **156** (2004) 175–221.
doi:10.1007/s00222-003-0337-0
- [TW] H. Thomas, N. Williams, Cyclic symmetry of the scaled simplex, preprint (2012).
arXiv:1207.5240 [math.CO]